Standing capillary-gravity waves of finite amplitude

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Standing surface waves in an inviscid incompressible fluid of finite depth are considered, taking into account the effect of capillary forces. Perturbation solutions for the surface profile, velocity potential, frequency of oscillation, and pressure are found to third order in the amplitude of the waves. A graph is given showing the regions in which the frequency of oscillation increases with amplitude and those in which it decreases with amplitude. These regions are defined as a function of the depth of the fluid and a parameter called the relative capillarity. A graph is also given showing the surface profile of a wave.

1. Introduction

The problem of standing gravity waves of finite amplitude on the surface of a fluid of uniform finite depth has been solved to third order by Tadjbakhsh & Keller (1960). (They are hereafter referred to as T & K.) The present paper applies their method to solve the more general problem, which includes capillary as well as gravitational forces. For long wavelengths under normal terrestrial conditions, capillary forces are generally negligible in comparison with gravitational forces. However, for short wavelengths or in an environment in which the acceleration field is less than the gravitational field of the earth at its surface, the capillary forces may no longer be negligible. The fluids considered here may have any surface tension and may be in an acceleration field of any magnitude that acts vertically downward, the only restriction being that the surface tension and acceleration field are not both zero, although one of them may be zero if the other is not.

An attempt is made to follow the notation of T & K so that their results can be easily compared with the ones presented here. Because of the introduction of surface tension and an acceleration field of arbitrary magnitude, a slightly different definition of some of the non-dimensional variables is required. However, in the absence of capillary forces and under normal terrestrial conditions, the variables reduce to theirs. Some of the details of their formulation are duplicated here for completeness. Of particular interest is the effect which introduction of capillary forces has on the critical depth found by T & K, at less than which the frequency of a wave increases with amplitude and at greater than which the frequency decreases with amplitude.

2. Formulation

The time-periodic, irrotational, two-dimensional motion of an inviscid incompressible fluid bounded below by a rigid horizontal plane and above by a free surface is considered. A uniform acceleration field of arbitrary strength acts vertically downward on the fluid, and surface tension effects are included. The motion is taken to be periodic in the horizontal direction and symmetric about the vertical plane $x = 0$, so that only the fluid between that plane and a parallel plane one-half wavelength from it need be considered. Let λ denote the wavelength; $k = 2\pi/\lambda$ the wave-number; $k^{-1}h$ the mean depth of the liquid; $k^{-1}x$ and $k^{-1}y$ the distances along the horizontal and vertical axes, respectively; κg the magnitude of the downward-acting uniform acceleration field, where g is the acceleration due to gravity and κ may be any non-negative number; and $\gamma = \sigma k^2/\rho g$, a dimensionless parameter proportional to Laplace's capillary constant, where σ is the surface tension of the liquid-vapour interface and ρ is the density of the liquid. Let $\delta = \gamma/(\kappa+\gamma)$ be a parameter called the relative capillarity; its value lies between zero and one. For $\delta \ll 1$, the capillary effects are small; whereas, for $(1-\delta) \ll 1$, they predominate. Finally, let $[kg(\kappa+\gamma)]^{\frac{1}{2}}\,\omega$ denote the angular frequency; $[kg(\kappa+\gamma)]^{-\frac{1}{2}}\,\omega^{-1}t$ the time; a the amplitude of the linearized surface wave motion; $\epsilon k^{-1}\eta(x,t)$ the elevation of the surface above the mean level given by the plane $y = 0$, and $\epsilon[g(\kappa+\gamma)]^{\frac{1}{2}}\,k^{-\frac{3}{2}}\phi(x,y,t)$ the velocity potential.

In terms of these dimensionless quantities, the equations of motion are

$$\Delta\phi = 0 \quad \text{in} \quad 0 < x < \pi \quad \text{and} \quad -h < y < \epsilon\eta(x,t), \tag{1}$$

$$(1-\delta)\,\eta - \delta[\eta_{xx}\{1-\tfrac{3}{2}\epsilon^2\eta_x^2 + O(\epsilon^3)\}] + \omega\phi_t + \tfrac{1}{2}\epsilon(\phi_x^2+\phi_y^2) = 0 \quad \text{on} \quad y = \epsilon\eta(x,t), \tag{2}$$

$$\phi_y = \omega\eta_t + \epsilon\phi_x\eta_x \quad \text{on} \quad y = \epsilon\eta(x,t), \tag{3}$$

$$\partial\phi/\partial n = 0 \quad \text{on} \quad x = 0,\ x = \pi,\ y = -h, \tag{4}$$

$$\eta_x = 0 \quad \text{on} \quad x = 0,\ x = \pi, \tag{5}$$

$$\int_0^\pi \eta(x,t)\,dx = 0, \tag{6}$$

$$\nabla\phi(x,y,t+2\pi) = \nabla\phi(x,y,t), \tag{7}$$

$$\int_{-h}^0 \int_0^\pi \int_0^{2\pi} \phi(x,y,t)\sin t\,\cos x\,dt\,dx\,dy = 0, \tag{8}$$

and

$$\int_{-h}^0 \int_0^\pi \int_0^{2\pi} \phi(x,y,t)\cos t\,\cos x\,dt\,dx\,dy = \tfrac{1}{2}\pi^2(\tanh h)^{\frac{1}{2}}. \tag{9}$$

Equation (1) is the Laplace equation governing irrotational flow; (2) is Bernoulli's law for constant external pressure at the free surface of the fluid including the Taylor's series expansion of the surface-tension terms to third order about $\epsilon = 0$; (3) is the condition that a particle once on the surface remain on the surface; (4) is the condition that the normal velocity component vanish on the planes of symmetry, $x = 0$ and $x = \pi$, and on the bottom rigid surface, $y = -h$; (5) is the condition that the slope of the free surface be continuous at $x = 0$ and $x = \pi$ if these are planes of symmetry, or that the contact angle be fixed at $\frac{1}{2}\pi$ during the motion if these are rigid bounding walls; (6) is the condition that the mean free surface is $y = 0$; (7) is the condition that the motion be periodic in time; and (8) and (9) fix the phase and amplitude of the motion.†

The pressure $p(x, y, t)$ is given by Bernoulli's law,

$$k(p-p_0)/\rho g(\kappa+\gamma) = -(1-\delta)\,y - \epsilon\omega\phi_t - \tfrac{1}{2}\epsilon^2(\phi_x^2+\phi_y^2), \tag{10}$$

where p_0 denotes the pressure of the atmosphere above the fluid. Because of the surface-tension effects, the pressure in the fluid just below the free surface $y = \epsilon\eta$ is not p_0, but, as combining (10) with (2) shows, is discontinuous by an amount

$$k(p-p_0)/\rho g(\kappa+\gamma) = -\epsilon\delta[\eta_{xx}\{1-\tfrac{3}{2}\epsilon^2\eta_x^2+O(\epsilon^3)\}] \quad \text{on} \quad y = \epsilon\eta. \tag{11}$$

The problem to be solved is the determination of $\eta(x,t)$, $\phi(x,y,t)$, and ω satisfying equations (1) through (9). This is done by determining the first three terms in the expansion of the solution in powers of ϵ.

As was noted by T & K for the problem without capillary effects, a unique solution does not exist for those values of h for which the linear theory yields a frequency that is an integral multiple of the fundamental frequency. The same holds true for the present problem, and in order to make the solution unique (except for the arbitrary additive constant to ϕ), it must be required that the frequency of the nth spatial harmonic $[n\{1+\delta(n^2-1)\}\tanh nh]^{\frac{1}{2}}$ is not an integral multiple of the fundamental frequency $(\tanh h)^{\frac{1}{2}}$. Thus the condition

$$\frac{n[1+\delta(n^2-1)]\tanh nh}{\tanh h} \neq j^2, \quad \text{for} \quad \begin{Bmatrix} n = 2, 3, \ldots, \\ j = 1, 2, \ldots, \end{Bmatrix} \tag{12}$$

is imposed.

3. Solution

The zero-order equations are found by assuming that η, ϕ, and ω have limits η^0, ϕ^0, and ω_0 as ϵ tends to zero. Conditions (2) and (3) then become

$$(1-\delta)\,\eta^0 - \delta\eta_{xx}^0 + \omega_0\phi_t^0 = 0 \quad \text{on} \quad y = 0, \tag{2.0}$$

$$\phi_y^0 - \omega_0\eta_t^0 = 0 \quad \text{on} \quad y = 0. \tag{3.0}$$

Equations (1) and (4) to (9) remain unchanged in form as equations in η^0, ϕ^0, and ω_0. The solution to the zero-order problem is

$$\eta^0 = \sin t \cos x, \tag{13}$$

$$\phi^0 = (\omega_0/\sinh h)\cos t \cos x \cosh(y+h), \tag{14}$$

$$\omega_0^2 = \tanh h. \tag{15}$$

Notice that the shape of the wave does not depend on the value of δ, the relative capillarity, so that the waveform obtained here is the same as the waveform obtained for the linear problem in the absence of surface tension. However, the frequency of oscillation is, in general, different, since the definition of the

† Dr Tadjbakhsh has pointed out to me that it is necessary to put the amplitude condition on ϕ rather than η so that the expansion parameter ϵ agrees with that of Penney & Price (1952). It is algebraically more convenient to put the phase condition on ϕ also, rather than on η as done by T & K.

dimensionless ω_0 depends upon the surface tension and magnitude of the acceleration field.

The first-order equations are found by assuming that η, ϕ, and ω have first derivatives with respect to ϵ at $\epsilon = 0$, where these derivatives are denoted by η^1, ϕ^1, and ω_1. Differentiating (1) to (9) with respect to ϵ, utilizing

$$d\phi(x, \epsilon\eta, t, \epsilon)/d\epsilon = [\partial/\partial\epsilon + (\eta + \epsilon\eta_\epsilon)\,\partial/\partial y]\,\phi$$

in (2) and (3), and letting $\epsilon = 0$ yields

$$(1-\delta)\,\eta^1 - \delta\eta^1_{xx} + \omega_0\,\phi^1_t = -\tfrac{1}{2}[(\phi^0_x)^2 + (\phi^0_y)^2] - \omega_0\,\eta^0\phi^0_{ty} - \omega_1\,\phi^0_t \quad \text{on} \quad y = 0, \tag{2.1}$$

$$\phi^1_y - \omega_0\,\eta^1_t = \eta^0_x\,\phi^0_x - \eta^0\phi^0_{yy} + \omega_1\,\eta^0_t \quad \text{on} \quad y = 0, \tag{3.1}$$

and

$$\int_{-h}^{0}\int_{0}^{\pi}\int_{0}^{2\pi} \phi^1(x,y,t)\cos t\,\cos x\,dt\,dx\,dy = 0. \tag{9.1}$$

Equations (1) and (4) to (8) remain of the same form as equations in η^1, ϕ^1, and ω_1.

Substitution of (13) to (15) into (2.1) and (3.1) yields

$$(1-\delta)\,\eta^1 - \delta\eta^1_{xx} + \omega_0\,\phi^1_t = \tfrac{1}{8}[(\omega_0^2 - \omega_0^{-2}) + (\omega_0^2 + \omega_0^{-2})\cos 2x - (3\omega_0^2 + \omega_0^{-2})$$

$$\times \cos 2t - (3\omega_0^2 - \omega_0^{-2})\cos 2t\cos 2x] + (\omega_1/\omega_0)\sin t\cos x \quad \text{on} \quad y = 0, \tag{16}$$

and

$$\phi^1_y - \omega_0\,\eta^1_t = -(1/2\omega_0)\sin 2t\,\cos 2x + \omega_1\cos t\cos x \quad \text{on} \quad y = 0. \tag{17}$$

Differentiation of (16) with respect to t and substitution of η^1_t from (17) and η^1_{xxt} from (17), which has been differentiated twice with respect to x, yields

$$-\delta\phi^1_{yxx} + (1-\delta)\,\phi^1_y + \omega_0^2\,\phi^1_{tt} = \tfrac{1}{4}(3\omega_0^3 + \omega_0^{-1})\sin 2t + \tfrac{3}{4}[\omega_0^3 - (1+2\delta)\,\omega_0^{-1}]$$

$$\times \sin 2t\,\cos 2x + 2\omega_1\cos t\,\cos x \quad \text{on} \quad y = 0. \tag{18}$$

Separation of variables yields for the solution of (1), subject to (4),

$$\phi^1(x,y,t) = \sum_{n=0}^{\infty} A_n(t)\cos nx\cosh n(y+h). \tag{19}$$

Substitution of (19) into (18) yields

$$\omega_0^2\,A_{0_{tt}} = \tfrac{1}{4}(3\omega_0^3 + \omega_0^{-1})\sin 2t, \tag{20}$$

$$\omega_0^2\cosh hA_{1_{tt}} + \sinh h\,A_1 = 2\omega_1\cos t, \tag{21}$$

$$\omega_0^2\cosh 2hA_{2_{tt}} + 2(1+3\delta)\sinh 2hA_2 = \tfrac{3}{4}[\omega_0^3 - (1+2\delta)\,\omega_0^{-1}]\sin 2t, \tag{22}$$

$$\omega_0^2\cosh nhA_{n_{tt}} + n[1 + (n^2-1)\,\delta]\sinh nhA_n = 0 \quad \text{for} \quad n = 3, 4, \ldots. \tag{23}$$

From (7) and (21), it follows that A_n must be periodic in t with period 2π for $n \geqslant 1$, and from (12) and (23) that $A_n = 0$ for $n \geqslant 3$. From (12), (15) and (22) there results

$$A_2 = -\frac{3[\omega_0 - 2\delta\omega_0^{-3} - (1+2\delta)\,\omega_0^{-7}]}{16(1 - 3\delta\omega_0^{-4})\cosh 2h}\sin 2t. \tag{24}$$

The periodicity of A_1 requires $\omega_1 = 0$, so that (21), (19), (8) and (9.1) then yield $A_1 = 0$. Finally, (20) yields

$$A_0 = -\tfrac{1}{16}(3\omega_0 + \omega_0^{-3})\sin 2t + \alpha_0\,t + \beta_0, \tag{25}$$

where α_0 and β_0 are constants to be determined.

Substitution of the above results into (19), and substitution of the resulting expression for ϕ^1 into (16), yields

$$-\delta\eta^1_{xx}+(1-\delta)\eta^1 = \tfrac{1}{8}(\omega_0^2-\omega_0^{-2})-\omega_0\alpha_0+\tfrac{1}{8}(\omega_0^2+\omega_0^{-2})\cos 2x + \tfrac{1}{8}[(1+3\delta)(\omega_0^{-2}-3\omega_0^{-6})/(1-3\delta\omega_0^{-4})]\cos 2t\cos 2x. \quad (26)$$

The solution of (26) subject to (5) is

$$\eta^1 = \frac{1}{8(1-\delta)}(\omega_0^2-\omega_0^{-2})-\frac{\omega_0\alpha_0}{1-\delta}+\frac{\omega_0^2+\omega_0^{-2}}{8(1+3\delta)}\cos 2x+\frac{\omega_0^{-2}-3\omega_0^{-6}}{8(1-3\delta\omega_0^{-4})}\cos 2t\cos 2x.$$

Equation (6) requires

$$\alpha_0 = \tfrac{1}{8}(\omega_0-\omega_0^{-3}).$$

The solution to the first-order problem is thus

$$\eta^1 = \frac{1}{8}\left[\frac{\omega_0^2+\omega_0^{-2}}{1+3\delta}+\frac{\omega_0^{-2}-3\omega_0^{-6}}{1-3\delta\omega_0^{-4}}\cos 2t\right]\cos 2x, \quad (27)$$

$$\phi^1 = \beta_0+\tfrac{1}{8}(\omega_0-\omega_0^{-3})t-\tfrac{1}{16}(3\omega_0+\omega_0^{-3})\sin 2t - \{3[\omega_0-2\delta\omega_0^{-3}-(1+2\delta)\omega_0^{-7}]/16(1-3\delta\omega_0^{-4})\cosh 2h\} \times \sin 2t\cos 2x\cosh 2(y+h), \quad (28)$$

$$\omega_1 = 0, \quad (29)$$

where β_0 is an arbitrary constant.

Notice that the first-order waveform depends upon the value of δ, so that the presence of capillary forces changes the shape of the wave from that for $\delta = 0$. In either case, $\omega_1 = 0$, however.

The second-order equations are found by assuming that η, ϕ, and ω have second derivatives with respect to ϵ at $\epsilon = 0$, where these derivatives are denoted by η^2, ϕ^2, and ω_2. Differentiating (1) to (9) twice with respect to ϵ and letting $\epsilon = 0$ leaves equations (1) and (4) to (8) unchanged in form as equations in η^2, ϕ^2, and ω_2. Equation (9) becomes of the same form as (9.1), and new equations (2.2) and (3.2) result giving the appropriate conditions on $y = 0$. Proceeding exactly as in the first-order case, elimination of η^2 from (2.2) and (3.2) yields

$$-\delta\phi^2_{yxx}+(1-\delta)\phi^2_y+\omega_0^2\phi^2_{tt} = \alpha_{11}\cos t\cos x+\alpha_{13}\cos t\cos 3x + \alpha_{31}\cos 3t\cos x+\alpha_{33}\cos 3t\cos 3x, \quad (30)$$

where

$$\left.\begin{aligned}
\alpha_{11} &= 2\omega_2+\frac{1}{16}\left[\frac{2\omega_0^5+3(1+9\delta^2)\omega_0+3(4+6\delta-9\delta^2-27\delta^3)\omega_0^{-3}-9(1+5\delta+4\delta^2)\omega_0^{-7}}{(1+3\delta)(1-3\delta\omega_0^{-4})}\right],\\
\alpha_{13} &= \frac{1}{16}\left[\frac{2\omega_0^5+(-5-18\delta+117\delta^2)\omega_0+3(-14\delta+9\delta^2-45\delta^3)\omega_0^{-3}+3(1+5\delta-12\delta^2-144\delta^3)\omega_0^{-7}}{(1+3\delta)(1-3\delta\omega_0^{-4})}\right],\\
\alpha_{31} &= \frac{1}{16}\left[\frac{(31-9\delta)\omega_0+(-62-24\delta+27\delta^2)\omega_0^{-3}-3(3+4\delta)\omega_0^{-7}}{(1-3\delta\omega_0^{-4})}\right],\\
&\text{and}\\
\alpha_{33} &= \frac{3}{16}\left[\frac{13(1-\delta)\omega_0-(22+32\delta-15\delta^2)\omega_0^{-3}+3(3+20\delta+16\delta^2)\omega_0^{-7}}{(1-3\delta\omega_0^{-4})}\right].
\end{aligned}\right\} \quad (31)$$

Solving for ϕ^2 and η^2 exactly as before (α_{11} is found to be zero, thus determining ω_2) yields the solution to the second-order problem as

$$\eta^2 = b_{11} \sin t \cos x + b_{13} \sin t \cos 3x + b_{31} \sin 3t \cos x + b_{33} \sin 3t \cos 3x, \tag{32}$$

$$\phi^2 = \beta_2 + \beta_{13} \cos t \cos 3x \cosh 3(y+h) + \beta_{31} \cos 3t \cos x \cosh (y+h) + \beta_{33} \cos 3t \cos 3x \cosh 3(y+h), \tag{33}$$

$$\omega_2 = \frac{1}{32}\left[\frac{-2\omega_0^5 - 3(1+9\delta^2)\,\omega_0 - 3(4+6\delta-9\delta^2-27\delta^3)\,\omega_0^{-3} + 9(1+5\delta+4\delta^2)\,\omega_0^{-7}}{(1+36)\,(1-3\delta\omega_0^{-4})}\right], \tag{34}$$

where β_2 is an arbitrary constant, and

$$b_{11} = \frac{1}{32}\left[\frac{2\omega_0^4 - (5+12\delta-27\delta^2) + 3(2+10\delta-3\delta^2-27\delta^3)\,\omega_0^{-4} + 3(1+5\delta)\,\omega_0^{-8}}{(1+3\delta)\,(1-3\delta\omega_0^{-4})}\right],$$

$$b_{13} = \frac{3}{128}\left[\frac{2\omega_0^8 + (1-18\delta-27\delta^2)\,\omega_0^4 - 3(5+24\delta+18\delta^2-27\delta^3) + 3(9+35\delta+39\delta^2+81\delta^3)\,\omega_0^{-4} + 9(1+5\delta+4\delta^2)\,\omega_0^{-8}}{(1+3\delta)\,(1-3\delta\omega_0^{-4})[1+3\delta(\omega_0^4+3)]}\right],$$

$$b_{31} = \frac{1}{128}\left[\frac{-5+3\delta+9(2-\delta^2)\,\omega_0^{-4} + 3(1-4\delta)\,\omega_0^{-8}}{(1-3\delta\omega_0^{-4})}\right],$$

$$b_{33} = \frac{3}{128}\,\frac{1-\delta+3(-1-\delta+\delta^2)\,\omega_0^{-4} + (3+4\delta+9\delta^2)\,\omega_0^{-8} - 3(3+4\delta)\,\omega_0^{-12}}{(1-3\delta\omega_0^{-4})\,[1-1+3\omega_0^{-4})]}, \tag{35}$$

and

$$\beta_{13} = \frac{1+3\omega_0^4}{128\cosh 3h}\left[\frac{2\omega_0^3 + (-5-18\delta+117\delta^2)\,\omega_0^{-1} + 3(-14\delta+9\delta^2-45\delta^3)\,\omega_0^{-5} + 3(1+5\delta-12\delta^2-144\delta^3)\,\omega_0^{-9}}{(1+36)\,(1-3\delta\omega_0^{-4})[1+3\delta(\omega_0^4+3)]}\right],$$

$$\beta_{31} = \frac{1}{128\cosh h}\left[\frac{(31-9\delta)\,\omega_0^{-1} + (-62-24\delta+27\delta^2)\,\omega_0^{-5} - 3(3+4\delta)\,\omega_0^{-9}}{1-3\delta\omega_0^{-4}}\right],$$

$$\beta_{33} = \frac{1+3\omega_0^4}{128\cosh 3h}\left[\frac{13(-1+\delta)\,\omega_0^{-5} + (22+32\delta-15\delta^2)\,\omega_0^{-9} - 3(3+20\delta+16\delta^2)\,\omega_0^{-13}}{(1-3\delta\omega_0^{-4})[1-\delta(1+3\omega_0^{-4})]}\right]. \tag{36}$$

Notice that the second-order waveform and second-order frequency both depend upon the value of δ, so that the presence of capillary forces changes them from their values for $\delta = 0$.

4. Conclusion

The final solution to the problem is found by substituting the results for the zero-, first- and second-order problems as given by (13) to (15), (27) to (29), and (32) to (34) into

$$\epsilon\eta = \epsilon\eta^0(x,t) + \epsilon^2\eta^1(x,t) + \tfrac{1}{2}\epsilon^3\eta^2(x,t) + O(\epsilon^4), \tag{37}$$

$$\epsilon\phi = \epsilon\phi^0(x,y,t) + \epsilon^2\phi^1(x,y,t) + \tfrac{1}{2}\epsilon^3\phi^2(x,y,t) + O(\epsilon^4), \tag{38}$$

and

$$\omega = \omega_0 + \tfrac{1}{2}\epsilon^2\omega_2 + O(\epsilon^3). \tag{39}$$

The pressure may then be found by substituting the appropriate derivatives of ϕ as calculated from (38) into (10).

Of significant interest is the variation of the frequency of oscillation with amplitude as given by (39). The difference between the frequency of oscillation ω and the fundamental frequency ω_0 is given to the desired order of approximation by the term $\frac{1}{2}\epsilon^2\omega_2$. Examination of (34) shows that ω_2 may be either positive or negative depending upon the values of ω_0 and δ. These quantities both lie between zero and one, ω_0 being determined by the mean depth of the liquid from (15). As h increases from zero to infinity, ω_0 increases from zero to one.

The regions of positive and negative ω_2 are shown in figure 1. To the left of the curve labelled I and to the right of the curve labelled II ω_2 is negative, and between the curves it is positive. Curve III is explained later. Curve I corresponds to a

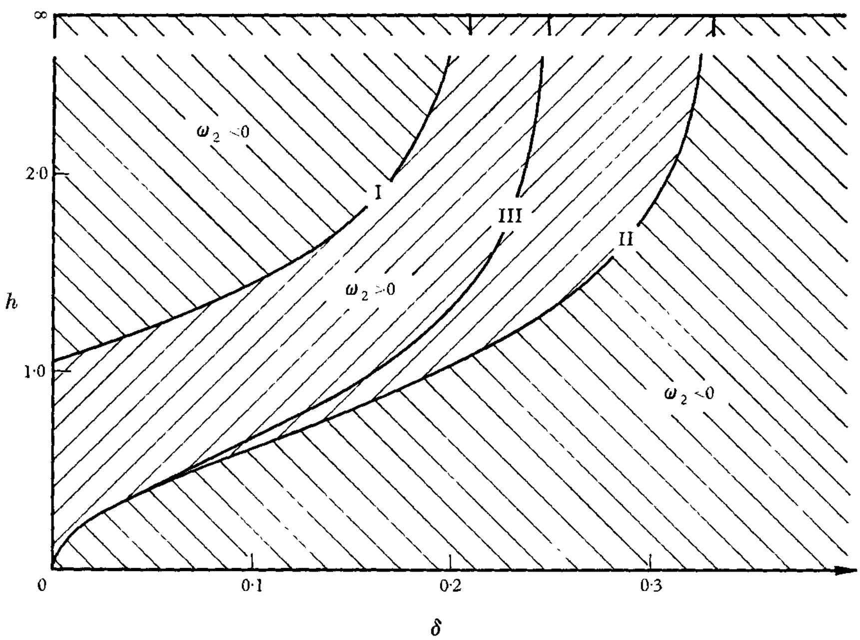

FIGURE 1. The location of the zeros and poles of ω_2 and the poles of η^1, ϕ^1, η^2, and ϕ^2. ω_2 is zero along I; ω_2, η^1, ϕ^1, η^2, and ϕ^2 have poles along II; η^2 and ϕ^2 have poles along III.

sign change in the numerator of (34), so that for values of δ and h lying on this curve ω_2 is zero. The intersection of this curve with the h axis at $h = 1{\cdot}06$ corresponds to the critical depth h^* found by T & K for $\delta = 0$.

Curve II corresponds to the sign change of the term $(1-3\delta\omega_0^{-4})$ in the denominator of (34). For values of δ and h lying on this curve, the denominator in the expression for ω_2 is zero, which represents a resonance condition for the second harmonic. Curve II, however, is the curve represented by (12) for $n = j = 2$, so that points on it are excluded by the uniqueness condition. For points near the curve, the coefficient of the second harmonic in the solutions for η and ϕ can still become very large.

Curve III corresponds to the sign change of the term $[1-\delta(1+3\omega_0^{-4})]$ in the denominator of b_{33} in (35) and β_{33} in (36), and represents a resonance condition for the third harmonic. This curve is given by (12) for $n = j = 3$; hence, points on it are excluded, but for points near it the solutions for η and ϕ contain large amounts of the third harmonic.

If the solution were carried out to higher order in ϵ, one would find additional resonance curves for the other harmonics and these curves would correspond to (12) for certain pairs of values of n and j. Points on these curves would thus be excluded, but for points near the curves, the amount of the corresponding harmonic present in η and ϕ would be large. These resonance curves all lie to the left of II, the higher the harmonic the closer the curve lies to the h-axis.

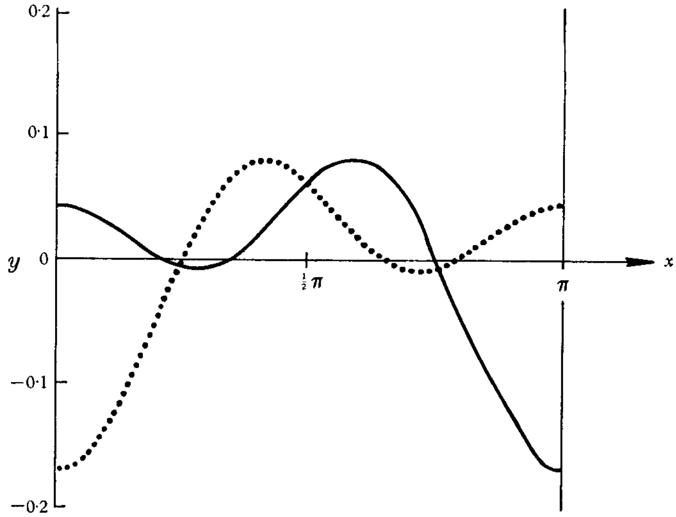

FIGURE 2. Standing-wave profile at $t = (n+\frac{1}{2})\pi$ for $\epsilon = 0{\cdot}05$, $h = 0{\cdot}25$, $\delta = 0{\cdot}04$. Solid curve is for n even and broken curve for n odd.

It should be understood, then, that equations (37), (38), and (39) form a solution to the problem in the sense that as ϵ approaches zero, the behaviour is as given. One does not imply, however, that for a given ϵ the low-order terms presented in (37), (38), and (39) are always larger than the additional terms one would obtain by carrying the solution out to higher orders in ϵ. Also, one could not use the solution for points too close to curves II and III, since too large a second or third harmonic would violate some of the implicit conditions of the problem such as the requirement that the lower bounding surface never be exposed or the requirement that the frequency of oscillation be positive.

In figure 2, the profile of one-half wavelength of the surface is shown as calculated from (37) at the times $t = (n+\frac{1}{2})\pi$, which correspond to the times when the velocity throughout the fluid is zero. These are the times when, for a given x, the surface is at either its highest or lowest position. The solid portion is for n odd and the dotted portion for n even, the surface oscillating between the two. The curves are calculated for $\epsilon = 0{\cdot}05$, $h = 0{\cdot}25$, and $\delta = 0{\cdot}04$. Figure 1 shows that $h = 0{\cdot}25$, $\delta = 0{\cdot}04$ is about the same distance from resonance curves II and III as is $h = 0{\cdot}25$, $\delta = 0$, so that the higher-order terms should be of about the same magnitude in each case, but generally different in sign. The curves for $h = 0{\cdot}25$ and $\delta = 0$ are given in figure 1 of T & K and comparison with their curves shows this to be so. The surface profile for values of h and δ farther away from curves II or III would contain less of the second and third harmonics and be composed primarily of the fundamental curve η^0 predicted by the linear theory.

Some of the effects discussed here should not be too difficult to observe in the laboratory. For example, a fluid depth $h = 0{\cdot}25$ corresponds approximately to $\delta = 0{\cdot}02$ on curves II and III in figure 1. Under normal terrestrial conditions, a value of $\delta = 0{\cdot}02$ is equivalent to a wavelength of about 10 cm in water. To achieve larger values of δ for reasonable wavelengths, however, one would have to experiment in a significantly reduced gravitational field.

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